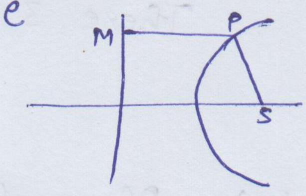
Model Answer

AV-8875

B.Sc. (Hon's) (Third Semester) Examination, 2015-16

Mathematics Paper - Second (Geometry)

1. (i) Conic : Locus of a point P such that ratio of the distances of P from fix point S (focus) and from fixed line MZ (directrix) is constant (eccentricity) is called conic ie $\frac{PS}{PM} = e$



Eqn of Conic $\frac{l}{r} = 1 + e\cos\theta$

Equation of circle when pole lies on circumference of circle and also initial line is along the diameter is $r = 2a\cos\theta$

(ii) If θ be the angle between two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$

then $$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

(iii) We know that if α, β, γ be the angles which a line makes with positive direction of X, Y, Z axes then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$
$$1-\sin^2\alpha + 1-\sin^2\beta + 1-\sin^2\gamma = 1$$
$$3-(\sin^2\alpha+\sin^2\beta+\sin^2\gamma) = 1$$
$$\sin^2\alpha+\sin^2\beta+\sin^2\gamma = 2$$

(iv) Equation of straight line passing through (α, β, γ), having d.c. proportional to l, m, n is $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$

Distance of (1, 5, 7) from plane $3x+9y-4z+12=0$ is

$$\frac{3(1)+9(5)-4(7)+12}{\sqrt{3^2+9^2+4^2}} = \frac{3+45-28+12}{\sqrt{9+81+16}}$$

$$= \frac{32}{\sqrt{106}}$$

(V) Radical Axis: Radical planes of 3 spheres, taken two at a time intersect in a line, called radical line or radical axis of three spheres.

Let $S_1=0$, $S_2=0$, $S_3=0$ be 3 spheres.

Then Radical plane of $S_1=0$, $S_2=0$ is $S_1-S_2=0$

" " " $S_2=0$, $S_3=0$ is $S_2-S_3=0$

" " " $S_1=0$, $S_3=0$ is $S_1-S_3=0$

These 3 radical planes passes through line

$\boxed{S_1=S_2=S_3}$ which is Radical axis of $S_1=0$, $S_2=0$, $S_3=0$.

(VI) Plane of Contact at any point for sphere: Locus of point of contacts to the tangent planes through a given point is called plane of contact to given point for sphere.

Eq^n of Plane of contact at (α,β,γ) for sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

is $$\alpha x+\beta y+\gamma z+u(x+\alpha)+v(y+\beta)+w(z+\gamma)+d=0$$

(vii) Reciprocal Cone: Two cones such that each is the locus of normal drawn through origin to the tangent planes to the other, are called reciprocal cones.

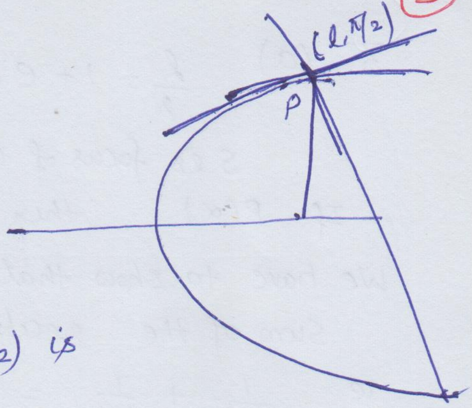
Reciprocal cone of cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

is $Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$ where

$A=bc-f^2$, $B=ac-g^2$, $C=ab-h^2$, $F=gh-af$, $G=hf-bg$, $H=fg-ch$

(viii) Equation of Hyperboloid on two sheet is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$



[2] Eq^n of normal to conic

$\frac{l}{r} = 1 + e\cos\theta$ is (at point α) ——— (1)

$$\frac{l}{r}\,\frac{e\sin\alpha}{1+e\cos\alpha} = e\sin\theta + \sin(\theta - \alpha)$$

Normal at one extremity of latus rectum $(l, \pi/2)$ is

$$\frac{l}{r}\cdot\frac{e\sin\pi/2}{1+e\cos\pi/2} = e\sin\theta + \sin(\theta - \pi/2)$$

$$\frac{el}{r} = e\sin\theta - \cos\theta \quad \text{———} \quad (2)$$

To find the distance from the focus of the other point in which normal (2) meets the conic (1) we have to solve (1) & (2).

From (1) $\cos\theta = \frac{1}{e}\left(\frac{l}{r} - 1\right)$, $\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \frac{1}{e^2}\left(\frac{l}{r}-1\right)^2}$

Put $\cos\theta$, $\sin\theta$ in (2)

$$\frac{el}{r} = e\sqrt{1 - \frac{1}{e^2}\left(\frac{l}{r}-1\right)^2} - \frac{1}{e}\left(\frac{l}{r} - 1\right)$$

$$\frac{el}{r} + \frac{1}{e}\left(\frac{l}{r} - 1\right) = e\sqrt{1 - \frac{1}{e^2}\left(\frac{l}{r}-1\right)^2}$$

Squaring on both sides

$$\frac{e^2l^2}{r^2} + \frac{1}{e^2}\left(\frac{l}{r}-1\right)^2 + \frac{2l}{r}\left(\frac{l}{r}-1\right) = e^2\left\{1 - \frac{1}{e^2}\left(\frac{l}{r}-1\right)^2\right\} = e^2 - \left(\frac{l}{r}-1\right)^2$$

$$e^2\left(\frac{l^2}{r^2} - 1\right) + \frac{1}{e^2}\left(\frac{l}{r}-1\right)^2 + \frac{2l}{r}\left(\frac{l}{r}-1\right) + \left(\frac{l}{r}-1\right)^2 = 0$$

$$\left(\frac{l}{r}-1\right)\left\{e^2\left(\frac{l}{r}+1\right) + \frac{1}{e^2}\left(\frac{l}{r}-1\right) + \frac{2l}{r} + \frac{l}{r} - 1\right\} = 0$$

If $\frac{l}{r} - 1 = 0 \Rightarrow r = l$ it shows point P which is not possible distance

then $e^2\left(\frac{l}{r}+1\right) + \frac{1}{e^2}\left(\frac{l}{r}-1\right) + \frac{3l}{r} - 1 = 0$

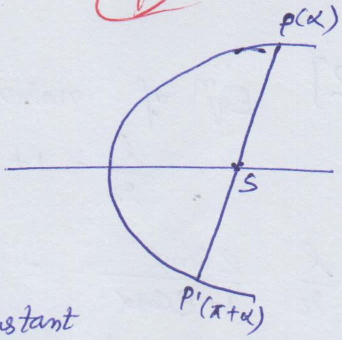
$$e^4 l + e^4 r + l - r + 3le^2 - e^2 r = 0$$

$$(e^4 + 1 + 3e^2)l - r(1 + e^2 - e^4) = 0$$

$$r = \frac{1 + 3e^2 + e^4}{1 + e^2 - e^4}\cdot l$$

Distance from the focus of the other point in which normal meets the conic is $\frac{1+3e^2+e^4}{1+e^2-e^4}\cdot l$

3. (a) $\frac{l}{r} = 1 + e\cos\theta$ —— ① is any Conic



S is focus of Conic. PSP' is any focal chord

If $P(\alpha)$ then $P'(\pi+\alpha)$

We have to show that

Sum of the reciprocals of segments of focal chord is Constant

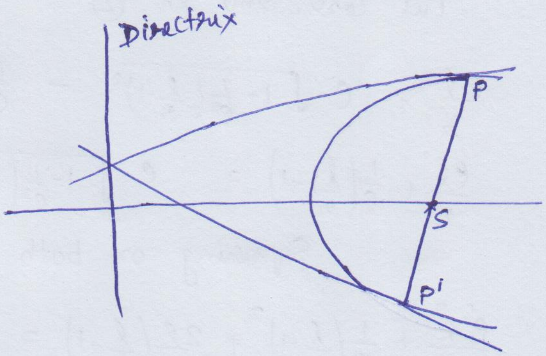
ie $\frac{1}{SP} + \frac{1}{SP'} = $ Constant

$\because P(SP, \alpha)$ lie on Conic ① $\Rightarrow \frac{l}{SP} = 1 + e\cos\alpha \Rightarrow SP = \frac{l}{1+e\cos\alpha}$

$P'(SP', \pi+\alpha)$ " " " $\Rightarrow \frac{l}{SP'} = 1 + e\cos(\pi+\alpha) \Rightarrow SP' = \frac{l}{1+e\cos(\pi+\alpha)} = \frac{l}{1-e\cos\alpha}$

Now $\frac{1}{SP} + \frac{1}{SP'} = \frac{1+e\cos\alpha}{l} + \frac{1-e\cos\alpha}{l}$

$\frac{1}{SP} + \frac{1}{SP'} = \frac{2}{l}$ = Constant



(b) PSP' is focal Chord of Conic

$\frac{l}{r} = 1 + e\cos\theta$ —— ①

Eqn of tangent at $P(\alpha)$ of Conic ① is

$\frac{l}{r} = e\cos\theta + \cos(\theta-\alpha)$ —— ②

Eqn of tangent at $P'(\pi+\alpha)$ of Conic ① is

$\frac{l}{r} = e\cos\theta + \cos(\theta - \overline{\pi+\alpha}) = e\cos\theta - \cos(\theta-\alpha)$ —— ③

For Intersection of tangents ② & ③

Adding ② & ③ $\frac{2l}{r} = 2e\cos\theta$

$\frac{l}{r} = e\cos\theta$ which is directrix of Conic ①

Showing that tangents at P and P' intersect on directrix.

4. (a) Eqn of tangent plane at (α, β, γ) to sphere $x^2+y^2+z^2=r^2$ is

$\alpha x + \beta y + \gamma z = r^2 \Rightarrow \frac{x}{r^2/\alpha} + \frac{y}{r^2/\beta} + \frac{z}{r^2/\gamma} = 1$

Compairing this eqn with intercept form of eqn of plane ie $x/a + y/b + z/c = 1$

we get $\frac{r^2}{\alpha} = a$, $\frac{r^2}{\beta} = b$, $\frac{r^2}{\gamma} = c$

$\Rightarrow \frac{1}{a^2} = \frac{\alpha^2}{r^4}$, $\frac{1}{b^2} = \frac{\beta^2}{r^4}$, $\frac{1}{c^2} = \frac{\gamma^2}{r^4}$

Adding we get $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{r^4}(\alpha^2+\beta^2+\gamma^2)$ —— ①

(α, β, γ) lie on given sphere $\Rightarrow \alpha^2+\beta^2+\gamma^2 = r^2$

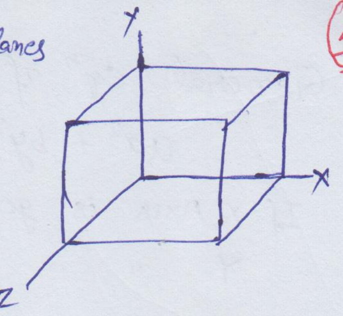
Put in ① we get

$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{r^4} \cdot r^2$

$\boxed{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{r^2}}$

Take centre of cube is origin, plane through centre ∥ to faces are coordinate planes

(b) Let Equation of 6 faces of cube is and each edge $=2a$.



$x=a, \quad x=-a, \quad y=a, \quad y=-a, \quad z=a, \quad z=-a$

Let (f,g,h) be any point

Distance of (f,g,h) from face $x=a$ is $\frac{f-a}{\sqrt{1}} = f-a$

" " " " " $x=-a$ is $f+a$

" " " " " $y=a$ is $g-a$

" " " " " $y=-a$ is $g+a$

" " " " " $z=a$ is $h-a$

" " " " $z=-a$ is $h+a$

From given conditions $(f-a)^2+(f+a)^2+(g-a)^2+(g+a)^2+(h-a)^2+(h+a)^2 = \text{Const.}$

$$2(f^2+g^2+h^2)+6a^2 = k$$

$$f^2+g^2+h^2 = \frac{k-6a^2}{2}$$

Locus of (f,g,h) is

$x^2+y^2+z^2 = \frac{k-6a^2}{2}$ which is a Sphere.

5. (a) We know that general equation of second degree

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$$

will represent cone if $\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0$

Now Equation $ax^2+by^2+cz^2+2ux+2vy+2wz+d=0$ represent cone

if $\begin{vmatrix} a & 0 & 0 & u \\ 0 & b & 0 & v \\ 0 & 0 & c & w \\ u & v & w & d \end{vmatrix} = 0$

$$a\begin{vmatrix} b & 0 & v \\ 0 & c & w \\ v & w & d \end{vmatrix} - u\begin{vmatrix} 0 & b & 0 \\ 0 & 0 & c \\ u & v & w \end{vmatrix} = 0$$

$$a\{b(cd-w^2)+v(-cv)\} - u\{-b(-uc)\} = 0$$

$$abcd - abw^2 - acv^2 - bcu^2 = 0$$

$$d - \frac{w^2}{c} - \frac{v^2}{b} - \frac{u^2}{a} = 0$$

$$\boxed{\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c} = d}$$

(b) General equation of a cone which touches the three coordinate planes is reciprocal cone of cone which has 3 coordinate axis as generators.

General eq^n of quadratic cone is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \text{——— (1)}$$

If X. Axis is generator $\Rightarrow (1,0,0)$ satisfies eq^n $\Rightarrow a = 0$

Y " " " $\Rightarrow (0,1,0)$ " " $\Rightarrow b = 0$

Z " " " $\Rightarrow (0,0,1)$ " " $\Rightarrow c = 0$

Put $a=0,\ b=0,\ c=0$ in (1) we get

$$2fyz + 2gzx + 2hxy = 0 \quad \text{——— (2)}$$

Now Reciprocal cone of above cone (2) is

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0 \quad \text{——— (3)}$$

where $A = bc - f^2 = -f^2$ $\quad$ ($\because$ For cone (2) $a=b=c=0$)

$B = ac - g^2 = -g^2$

$C = ab - h^2 = -h^2$

$F = gh - af = gh$

$G = hf - bg = hf$

$H = fg - ch = fg$

Put A, B, C, F, G, H in (3)

$$-f^2x^2 - g^2y^2 - h^2z^2 + 2ghyz + 2hfzx + 2fgxy = 0$$

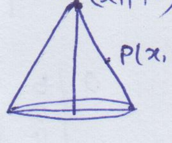
$$(fx + gy - hz)^2 - 4fgxy = 0$$

$$fx + gy - hz = \pm 2\sqrt{fgxy}$$

$$(\sqrt{fx} \pm \sqrt{gy})^2 = hz$$

$$\boxed{\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0}$$

6. (a) Eq^n of line through $A(\alpha,\beta,\gamma)$ is



$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \text{——— (1)}$$

Intersection of this line with $z = 0$

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{-\gamma}{n}$$

$$x = \alpha - \frac{l\gamma}{n}, \quad y = \beta - \frac{m\gamma}{n}$$

Intersection pt. is $\left(\alpha - \frac{l\gamma}{n},\ \beta - \frac{m\gamma}{n}\right)$

This pt lie on $y^2 = 4ax$ $\Rightarrow$ $\left(\beta - \frac{m\gamma}{n}\right)^2 = 4a\left(\alpha - \frac{l\gamma}{n}\right)$ ——— (2)

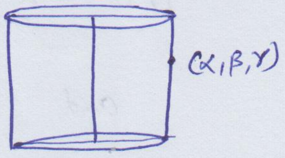
From (1) $\frac{l}{n} = \frac{x-\alpha}{z-\gamma}$, $\frac{m}{n} = \frac{y-\beta}{z-\gamma}$ Put in (2)

$$\left(\beta - \gamma \cdot \frac{y-\beta}{z-\gamma}\right)^2 = 4a\left(\alpha - \gamma \cdot \frac{x-\alpha}{z-\gamma}\right)$$

$$(\beta z - \cancel{\beta\gamma} - y\gamma + \cancel{\beta\gamma})^2 = 4a(\alpha z - \cancel{\alpha\gamma} - x\gamma + \cancel{\alpha\gamma})(z-\gamma)$$

$$\boxed{(\beta z - y\gamma)^2 = 4a(\alpha z - \gamma x)(z-\gamma)}$$

(b) Eq^n of Axis of cylinder $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$



Eq^n of generator through (α,β,γ) is

$$\frac{x-\alpha}{1} = \frac{y-\beta}{-2} = \frac{z-\gamma}{3}$$

Intersection with $z=0$ $\qquad \frac{x-\alpha}{1} = \frac{y-\beta}{-2} = \frac{-\gamma}{3}$

$$\Rightarrow \quad x = \alpha - \frac{\gamma}{3}, \quad y = \beta + \frac{2\gamma}{3}$$

Intersection pt. $\left(\alpha - \frac{\gamma}{3}, \beta + \frac{2\gamma}{3}\right)$

This point lie on $x^2+y^2=1$

if $\qquad \left(\alpha-\frac{\gamma}{3}\right)^2 + \left(\beta+\frac{2\gamma}{3}\right)^2 = 1$

Locus of (α,β,γ) is

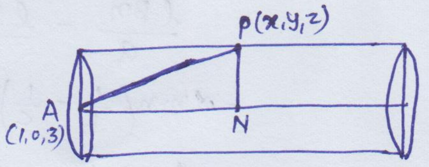
$$\left(x-\frac{z}{3}\right)^2 + \left(y+\frac{2z}{3}\right)^2 = 1$$

$$(3x-z)^2 + (3y+2z)^2 = 9$$

$$9x^2+z^2-6zx+9y^2+4z^2+12yz-9=0$$

$$\boxed{9x^2+9y^2+5z^2+12yz-6zx-9=0}$$ which is eq^n of cylinder.

7. Axis of cylinder: $\frac{x-1}{2} = \frac{y}{3} = \frac{z-3}{1}$



$P(x,y,z)$ is any pt on cylinder

Radius of Right Circular Cylinder $PN = 2$

$$AP = \sqrt{(x-1)^2+y^2+(z-3)^2}$$

AN = Projection of AP on AN

$$= \{(x-1)\hat{i} + y\hat{j} + (z-3)\hat{k}\} \cdot \left\{\frac{2}{\sqrt{14}}\hat{i} + \frac{3}{\sqrt{14}}\hat{j} + \frac{1}{\sqrt{14}}\hat{k}\right\}$$

$$= \frac{2(x-1)}{\sqrt{14}} + \frac{3y}{\sqrt{14}} + \frac{z-3}{\sqrt{14}}$$

In ΔPAN $\qquad AP^2 = AN^2 + PN^2$

$$(x-1)^2+y^2+(z-3)^2 = \frac{1}{14}(2x-2+3y+z-3)^2 + 4$$

$$14(x^2+1-2x+y^2+z^2+9-6z) = (2x+3y+z-5)^2 + 56$$

$$14x^2+14y^2+14z^2-28x-84z+140 = 4x^2+9y^2+12xy+z^2+25-10z+4zx+6zy-20x-30y+56=0$$

$$\boxed{10x^2+5y^2+13z^2-12xy-6yz-4zx-8x+30y-74z+59=0}$$

which is eq^n of right circular cylinder.

8. Let $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$ —— ① is straight line through (α, β, γ)

and this line ① is perpendicular to its polar line.

Now Polar line of line ① is given by

$$\left.\begin{array}{l} a\alpha x + b\beta y + c\gamma z = 1, \\ alx + bmy + cnz = 0 \end{array}\right\} \quad \text{—— ②}$$

If λ, μ, ν be the d.c. of this polar line then

$$\left.\begin{array}{l} a\alpha\lambda + b\beta\mu + c\gamma\nu = 0 \\ al\lambda + bm\mu + cn\nu = 0 \end{array}\right\}$$

Solving for $a\lambda, b\mu, c\nu$ we have

$$\frac{a\lambda}{\beta n - \gamma m} = \frac{b\mu}{\gamma l - \alpha n} = \frac{c\nu}{\alpha m - \beta l} = k$$

$$\Rightarrow \quad \lambda = \frac{k}{a}(\beta n - \gamma m), \quad \mu = \frac{k}{b}(\gamma l - \alpha n), \quad \nu = \frac{k}{c}(\alpha m - \beta l)$$

If line ① and line ② are perpendicular to each other

then $$l\lambda + m\mu + n\nu = 0$$

$$l\frac{k}{a}(\beta n - \gamma m) + m\frac{k}{b}(\gamma l - \alpha n) + n\frac{k}{c}(\alpha m - \beta l) = 0$$

$$\frac{l\beta n}{a} - \frac{l\gamma m}{a} + \frac{m\gamma l}{b} - \frac{m\alpha n}{b} + \frac{n\alpha m}{c} - \frac{n\beta l}{c} = 0$$

$$-\alpha mn\left(\frac{1}{b} - \frac{1}{c}\right) - \beta nl\left(\frac{1}{c} - \frac{1}{a}\right) - \gamma ml\left(\frac{1}{a} - \frac{1}{b}\right) = 0$$

$$\frac{\alpha}{l}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{\beta}{m}\left(\frac{1}{c} - \frac{1}{a}\right) + \frac{\gamma}{n}\left(\frac{1}{a} - \frac{1}{b}\right) = 0 \quad \text{—— ③}$$

Locus of ~~(α, β, γ)~~ straight line ① is given by eliminating l, m, n from ① & ③

From ① $\quad l = \frac{x-\alpha}{r}, \quad m = \frac{y-\beta}{r}, \quad n = \frac{z-\gamma}{r}$

Put in ③

$$\boxed{\frac{\alpha}{x-\alpha}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{\beta}{y-\beta}\left(\frac{1}{c} - \frac{1}{a}\right) + \frac{\gamma}{z-\gamma}\left(\frac{1}{a} - \frac{1}{b}\right) = 0}$$

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